Towards a virtual element approximation of single-layer and hypersingular boundary integral operators in acoustic scattering

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Introduction

GOAL. We investigate a novel Virtual Element-based discretization for boundary integral formulations of acoustic wave-scattering problems.

STATE OF THE ART. An analogous Virtual Element approximation-based Boundary Element Method (V-BEM) strategy was recently developed by E. Arcese, S. Pernet, and A. Touzalin[1] for electromagnetic scattering problems. We extend this framework to the acoustic setting.

MOTIVATION. Virtual Element Method (VEM) naturally accommodates general polytopal surface meshes — including non-matching interfaces and hanging nodes — without loss of accuracy, relaxing the meshing constraints of standard BEM discretizations.

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[1] E. Arcese, S. Pernet, and A. Touzalin. *A priori error analysis of a virtual element approximation-based boundary element method for the electric field integral equation*. To appear in *IMA Journal of Numerical Analysis* (2026). HAL Id: hal-04529074v2.

The model problem

Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary $\Gamma := \partial\Omega$. We consider the classical sound-hard acoustic scattering problem

$$\begin{cases} \Delta u + \kappa^2 u = 0, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \dfrac{\partial u}{\partial \mathbf{n}} = -\dfrac{\partial u^{\text{inc}}}{\partial \mathbf{n}}, & \text{on } \Gamma, \\ \lim\limits_{r \to 0} r(\partial_r - \imath\kappa)u = 0, & \end{cases}$$

where $\kappa \in \mathbb{R}_+$ is the wavenumber, $r = \|\mathbf{x}\|$ with $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{n}$ is the normal versor pointing outward of Γ. The function u is the unknown scattered field while u^{inc} is the known incident field.

Integral representation

Recall the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$:

$$G_\kappa(\mathbf{x},\mathbf{y}) = \frac{e^{\imath\kappa\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|}.$$

We seek the scattered field u in $\mathbb{R}^3 \setminus \overline{\Omega}$ by means of the double layer potential:

$$u(\mathbf{x}) = -\int_\Gamma \frac{\partial G_\kappa}{\partial \mathbf{n}_\mathbf{y}}(\mathbf{x},\mathbf{y})\,\psi(\mathbf{y})\,d\mathbf{y}, \qquad \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega},$$

where $\psi \in H^{1/2}(\Gamma)$ is an unknown density.

Boundary integral equation

We define, for all $\mathbf{x} \in \Gamma$, the hypersingular operator

$$\mathrm{W}_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \quad \mathrm{W}_\kappa \psi(\mathbf{x}) := -\frac{\partial}{\partial \mathbf{n}_\mathbf{x}} \int_\Gamma \frac{\partial G_\kappa}{\partial \mathbf{n}_\mathbf{y}}(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y})\, d\mathbf{y}.$$

Thanks to the double layer representation formula, and by imposing the Neumann boundary condition, we get that the unknown density ψ satisfies

$$\mathrm{W}_\kappa \psi(\mathbf{x}) = -\frac{\partial u^{\mathsf{inc}}}{\partial \mathbf{n}}, \quad \forall \mathbf{x} \in \Gamma. \tag{1}$$

Continuous variational formulation

By testing the integral equation (1) against a function $\phi \in H^{1/2}(\Gamma)$ we get

Find $\psi \in H^{1/2}(\Gamma)$ such that

$$a(\psi, \phi) = -\left\langle \frac{\partial u^{\mathrm{inc}}}{\partial \mathbf{n}}, \phi \right\rangle_\Gamma = -\int_\Gamma \frac{\partial u^{\mathrm{inc}}}{\partial \mathbf{n}}(\mathbf{x})\, \phi(\mathbf{x})\, d\mathbf{x}, \quad \forall \phi \in H^{1/2}(\Gamma), \tag{2}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the $H^{-1/2} - H^{1/2}$ duality pairing, and

$$a \colon H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{C},$$

$$a(\psi, \phi) := \langle \mathrm{W}_\kappa \psi, \phi \rangle_\Gamma = -\int_\Gamma \frac{\partial}{\partial \mathbf{n}_\mathbf{x}} \int_\Gamma \frac{\partial G_\kappa}{\partial \mathbf{n}_\mathbf{y}}(\mathbf{x}, \mathbf{y})\, \psi(\mathbf{y})\, \phi(\mathbf{x})\, d\mathbf{y}\, d\mathbf{x}.$$

Equivalent formulation

First, let's define the single layer operator

$$\mathrm{V}_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \quad \mathrm{V}_\kappa\lambda(\mathbf{x}) := \int_\Gamma G_\kappa(\mathbf{x},\mathbf{y})\,\lambda(\mathbf{y})\,d\mathbf{y}.$$

By applying the well-known representation formula for the hypersingular operator[2], we can rewrite the bilinear form by means of the single layer operator:

$$\begin{aligned} a(\psi,\phi) &= \langle \mathrm{V}_\kappa \mathbf{curl}_\Gamma\psi, \mathbf{curl}_\Gamma\phi\rangle_\Gamma - \kappa^2\langle \mathrm{V}_\kappa(\mathbf{n}\psi), \mathbf{n}\phi\rangle_\Gamma \\ &= \int_\Gamma\int_\Gamma G_\kappa(\mathbf{x},\mathbf{y})\ \left(\mathbf{curl}_\Gamma\psi(\mathbf{y})\cdot\mathbf{curl}_\Gamma\phi(\mathbf{x})\right)\ d\mathbf{y}\,d\mathbf{x} \\ &\quad - \kappa^2\int_\Gamma\int_\Gamma G_\kappa(\mathbf{x},\mathbf{y})\,\psi(\mathbf{y})\,\phi(\mathbf{x})\,(\mathbf{n}_\mathbf{y}\cdot\mathbf{n}_\mathbf{x})\,d\mathbf{y}\,d\mathbf{x}. \end{aligned}$$

$$\mathbf{curl}_\Gamma\varphi := \mathbf{n}\times\nabla_\Gamma\varphi$$

[2] S. A. Sauter and C. Schwab, *Boundary Element Methods*, Springer-Verlag, Springer Series in Computational Mathematics 39, Berlin, 2011.

Well-posedness of the continuous problem

Denoting with W_0 the hypersingular operator associated to the Laplace operator, namely $G_0(\mathbf{x}, \mathbf{y}) = 1/(4\pi\|\mathbf{x}-\mathbf{y}\|)$, we have the following results.

Proposition

Let $\Gamma = \partial\Omega$ be a closed surface. Then

- *The operator W_κ is continuous;*
- *The operator $\mathrm{W}_\kappa - \mathrm{W}_0 : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact;*
- *Problem (2) is well-posed, as long as κ^2 is not a Neumann eigenvalue for $-\Delta$ in Ω.*

S. A. Sauter and C. Schwab, *Boundary Element Methods*, Springer-Verlag, Springer Series in Computational Mathematics 39, Berlin, 2011.

Discrete setting

Let $\{\mathcal{T}_h\}$ be a family of polygonal decompositions of Γ. Each element $E \in \mathcal{T}_h$ has diameter $h_E \leq h$. Moreover, we suppose that the mesh satisfies the usual assumptions required by the VEM[3].

Then, we can introduce the local lowest-order enhanced VEM space[3]

$$V_h^1(E) := \{v \in H^1(E) \colon v|_e \in \mathbb{P}_1(e),\ \forall e \subseteq \partial E,\ \Delta v \in \mathbb{P}_1(E),\ \Pi_1^{0,E} v = \Pi_1^{\nabla,E} v\}.$$

The degrees of freedom (dof) consist of the value at the vertices of E, so that $\dim V_h^1(E) = \#$vertices of E.

The global counterpart is obtained by glueing

$$V_h^1 := \{v \in H^1(\Gamma) \colon v|_E \in V_h^1(E)\}.$$

[3] B. Ahmad, A. Alsaedi, F. Brezzi, L. D. Marini, A. Russo, *Equivalent projectors for virtual element methods*, Comput. Math. Appl. 66 (2013), no. 3, 376–391.

Projection operators[3]

$\Pi_1^{0,E}: L^2(E) \to \mathbb{P}_1(E)$ is the L^2-projection operator, defined for any $v_h \in V_h^1(E)$ by

$$\int_E \left(\Pi_1^{0,E} v_h\right) q \, d\mathbf{x} = \int_E v_h \, q \, d\mathbf{x}, \quad \forall q \in \mathbb{P}_1(E).$$

$\Pi_1^{\nabla,E}: H^1(E) \to \mathbb{P}_1(E)$ is the H^1-projection operator, defined for any $v_h \in V_h^1(E)$ by

$$\begin{cases} \displaystyle\int_E \nabla \Pi_1^{\nabla,E} v_h \cdot \nabla q \, d\mathbf{x} = \underbrace{\int_E \nabla v_h \cdot \nabla q \, d\mathbf{x}}_{\text{Computable from dof}}, \quad \forall q \in \mathbb{P}_1(E), \\ \displaystyle\int_{\partial E} \Pi_1^{\nabla,E} v_h = \int_{\partial E} v_h. \end{cases}$$

The respective global operators are obtained by glueing, i.e. for any $v_h \in V_h^1$,

$$(\Pi_1^{\bullet} v_h)\,|_E = \Pi_1^{\bullet,E}(v_h|_E) \quad \bullet = 0, \nabla.$$

Discrete problem: V-BEM

The discrete variational formulation reads as follows

$$\text{Find } \psi_h \in V_h^1 \text{ such that}$$
$$a_h(\psi_h, \phi_h) = -\sum_{E\in\mathcal{T}_h} \int_E \frac{\partial u^{\text{inc}}}{\partial \mathbf{n}}(\mathbf{x})\, \Pi_1^{\nabla,E}[\phi_h](\mathbf{x})\, d\mathbf{x}, \quad \forall \phi_h \in V_h^1. \tag{3}$$

Where

$$a_h(\psi_h, \phi_h) = a(\Pi_1^{\nabla}\psi_h, \Pi_1^{\nabla}\phi_h) + S_h\left(\left(\mathrm{I} - \Pi_1^{\nabla}\right)\psi_h, \left(\mathrm{I} - \Pi_1^{\nabla}\right)\phi_h\right).$$

Notice that on conforming triangular meshes, where $\Pi_1^{\nabla} = \mathrm{I}$, this numerical method coincides with the classic Galerkin BEM.

Discrete bilinear form

More explicitly, we set the discretized global bilinear form as $a_h : V_h^1 \times V_h^1 \to \mathbb{C}$

$$a_h(\psi_h, \phi_h) = \int_\Gamma \int_\Gamma G_\kappa(\mathbf{x}, \mathbf{y}) \left(\widetilde{\mathbf{curl}}_\Gamma \left[\Pi_1^\nabla \psi_h(\mathbf{y}) \right] \right) \cdot \left(\widetilde{\mathbf{curl}}_\Gamma \left[\Pi_1^\nabla \phi_h(\mathbf{x}) \right] \right) \, d\mathbf{y} \, d\mathbf{x}$$
$$- \kappa^2 \int_\Gamma \int_\Gamma G_\kappa(\mathbf{x}, \mathbf{y}) \, \Pi_1^\nabla \psi_h(\mathbf{y}) \, \Pi_1^\nabla \phi_h(\mathbf{x}) \, (\mathbf{n}_\mathbf{x} \cdot \mathbf{n}_\mathbf{y}) \, d\mathbf{y} \, d\mathbf{x}$$
$$+ S_h((\mathrm{I} - \Pi_1^\nabla)\psi_h, (\mathrm{I} - \Pi_1^\nabla)\phi_h).$$

where $\widetilde{\mathbf{curl}}_\Gamma$ denotes the element-by-element $\mathbf{curl}_\Gamma$ and the stabilization term S_h is given by

$$S_h(u_h, v_h) = \sum_{E \in \mathcal{T}_h} h_E^{-1} S_h^E(u_h, v_h), \quad S_h^E(u_h, v_h) = \sum_{i=1}^{N_{\text{dof}}^E} \mathrm{dof}_i(u_h) \, \mathrm{dof}_i(v_h)$$

Discrete V-BEM inf-sup condition

Define the operator

$$\mathrm{T}_\kappa : H^{1/2}(\Gamma) \to H^{1/2+\varepsilon}(\Gamma), \quad \mathrm{T}_\kappa := -(\mathrm{W}_\kappa^*)^{-1}(\mathrm{W}_\kappa - \mathrm{W}_0 - \mathrm{V}_0),$$

for some $\varepsilon \in (0, 1/2)$, and let $\mathrm{P}_h : H^{1/2}(\Gamma) \to V_h^1$ the global orthogonal projection operator that minimizes the $H^{1/2}-$norm. By choosing as test function $\phi_h = (\mathrm{I} + \mathrm{P}_h\mathrm{T}_\kappa)\psi_h$, it's possible to prove

Proposition

There exists positive constants $c > 0$ and $h_0 > 0$ such that for any $h < h_0$,

$$\inf_{\substack{\psi_h \in V_h^1 \\ \psi_h \neq 0}} \sup_{\substack{\phi_h \in V_h^1 \\ \phi_h \neq 0}} \frac{\Re\{a_h(\psi_h, \phi_h)\}}{\|\psi_h\|_{H^{1/2}(\Gamma)}\|\phi_h\|_{H^{1/2}(\Gamma)}} \gtrsim c.$$

Numerical test

For our numerical tests, we consider the following problem.

$$\begin{cases} -\Delta u - \kappa^2 u = 0, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \dfrac{\partial u}{\partial \mathbf{n}} = -\dfrac{\partial}{\partial \mathbf{n}} \exp(\imath\kappa \mathbf{d}\cdot\mathbf{x}), & \text{on } \Gamma, \\ \lim\limits_{r\to 0} r(\partial_r - \imath\kappa)u = 0, \end{cases} \iff \mathrm{W}_\kappa \psi(\mathbf{x}) = -\frac{\partial}{\partial \mathbf{n}} \exp(\imath\kappa \mathbf{d}\cdot\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma.$$

where $\Omega = \{\mathbf{x} \in \mathbb{R}^3 \colon \|\mathbf{x}\| < 1\}$, $\kappa = 2\pi$ and $\mathbf{d} = (0,0,1)$. The exact density on Γ is known, and is

$$\psi(\mathbf{x}) = -\frac{\imath}{\kappa^2} \sum_{\ell=0}^{\infty} \imath^\ell (2\ell+1) \frac{P_\ell(x_3/\|\mathbf{x}\|)}{h_\ell^{(1)\prime}(\kappa\|\mathbf{x}\|)}, \quad \forall \mathbf{x} \in \Gamma = \{\|\mathbf{x}\| = 1\}.$$

On the exact density

In the code, we compute the exact density as

$$\psi(\mathbf{x}) = -\frac{\imath}{\kappa^2} \sum_{\ell=0}^{\text{LMAX}} \imath^\ell (2\ell+1) \frac{P_\ell(x_3)}{h_\ell^{(1)\prime}(\kappa)}, \quad \forall \mathbf{x} \in \Gamma,$$

where P_ℓ is the $\ell-$th Legendre polynomial

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_{n+1}(t) = \frac{2n+1}{n+1}\, t\, P_n(t) - \frac{n}{n+1} P_{n-1}(t), \quad \forall n = 1, 2, \ldots,$$

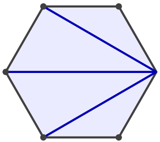
the function $h_\ell^{(1)}$ is the $\ell-$th spherical Hankel function of the first kind and LMAX is given by[4]

$$\text{LMAX} = \operatorname{floor}\left[\Re\{\kappa\} + 10 \ln(\Re\{\kappa\} + \pi)\right].$$

[4] R. Coifman, V. Rokhlin and S. Wandzura, *The fast multipole method for the wave equation: a pedestrian prescription*, IEEE Antennas and Propagation Magazine 35 (1993), no. 3, 7-12.

Computation of the integrals - 1

In the code, we have implemented two different techniques to compute the arising weakly singular integrals.
The first technique is based on the one proposed by Sauter-Schwab[5], which permits to analytically compute the singular integrals on a triangular domain, by means of a regularizing (Duffy) transformation. If instead we have a polygonal domain, we first triangulate the polygon and then we can apply the previous technique.

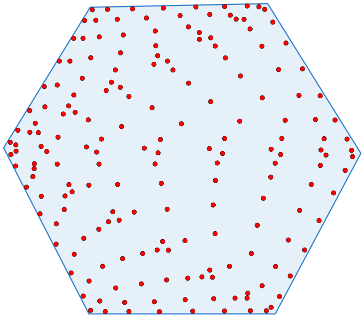


[5] S. A. Sauter and C. Schwab, *Boundary Element Methods*, Springer-Verlag, Springer Series in Computational Mathematics 39, Berlin, 2011.

Numerical computation of the integrals - 2

The second technique, which will be denoted in the numerical tests as `Mixed`, is based on two different contributions.

- For the cubature nodes for the outer integral, we employ Sommariva & Vianello's cubature rule[6], which are defined over general polygonal domains and characterized by a prescribed degree of exactness.



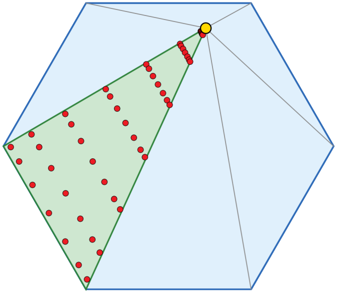
[6] B. Bauman, A. Sommariva, and M. Vianello, *Compressed algebraic cubature over polygons with applications to optical design*, J. Comput. Appl. Math., 370 (2020), 112658.

Numerical computation of the integrals - 2

The second technique, which will be denoted in the numerical tests as `Mixed`, is based on two different contributions.

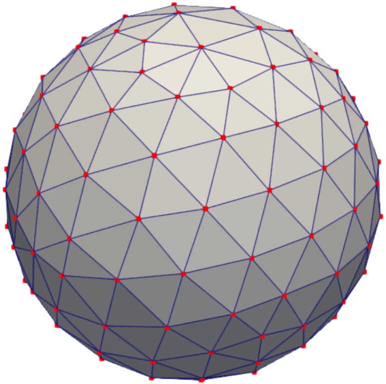
- Regarding the computation of the inner singular integral, i.e. when both variables of integration belong to the same polygonal element, we apply the following formula[7]

$$\int_{\widehat{E}} \frac{g(\widehat{\mathbf{y}})}{\|\widehat{\mathbf{x}}_P - \widehat{\mathbf{y}}\|}\, d\widehat{\mathbf{y}} = \sum_{i=1}^{n_E} \int_0^1 \int_0^1 g\left(\varphi(\xi, t)\right) \frac{(\mathbf{c}_i(t) - \widehat{\mathbf{x}}_P) \cdot \mathbf{c}_i'^{\perp}(t)}{\|\mathbf{c}_i(t) - \widehat{\mathbf{x}}_P\|}\, d\xi\, dt$$



[7] E. B. Chin and N. Sukumar, *Scaled boundary cubature scheme for numerical integration over planar regions with affine and curved boundaries*, Comput. Methods Appl. Mech. Eng., 380 (2021), 113796.

Triangular mesh of the sphere



$h[m]$	N_{dof}
5.05×10^{-1}	162
2.56×10^{-1}	642
1.28×10^{-1}	2562
6.41×10^{-2}	10242
3.21×10^{-2}	40962

Convergence behavior

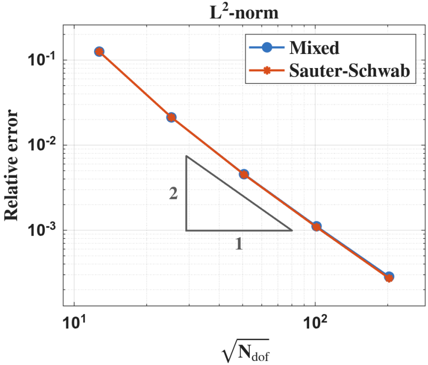


Figure: Convergence plot of the L^2 relative error.

$$\varepsilon_h = \frac{\|\psi - \Pi_1^{\nabla}\psi_h\|_{L^2(\Gamma)}}{\|\psi\|_{L^2(\Gamma)}}$$

Solution behavior

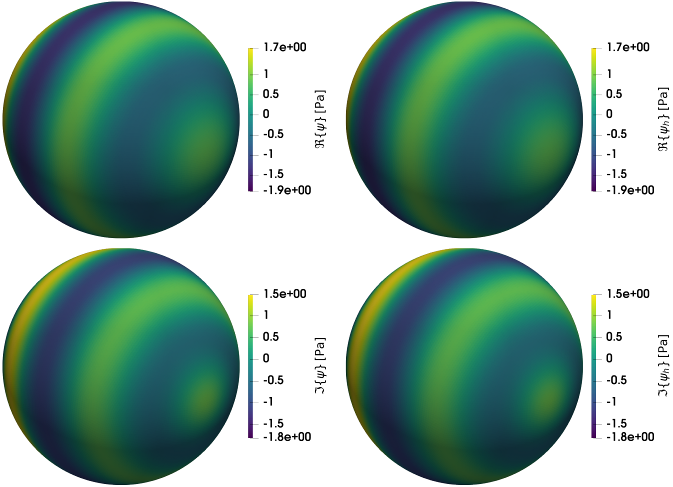


Figure: From right to left: exact and approximate density. From top to bottom: real and imaginary part.

Polygonal mesh of the sphere

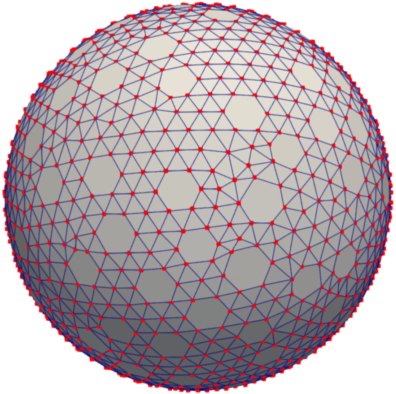


Figure: Polygonal mesh of the sphere.

$h[m]$	N_{dof}	Polygons/Elements
2.18×10^{-1}	1467	$123/2564$
1.75×10^{-1}	2499	$208/4361$
1.34×10^{-1}	3990	$333/6969$
1.16×10^{-1}	5604	$467/9794$
8.35×10^{-2}	9937	$828/17367$
7.64×10^{-2}	12664	$1055/22143$

The meshes are not nested.

Convergence behavior

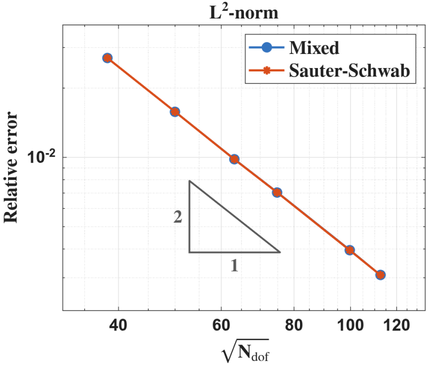


Figure: Convergence behavior of the L^2 relative error.

$$\text{EOC} = \frac{\ln\left(\varepsilon_{h_i}/\varepsilon_{h_{i+1}}\right)}{\ln\left(\sqrt{N_{\text{dof}}^{i+1}/N_{\text{dof}}^{i}}\right)}$$

N_{dof}	EOC
1467	—
2499	2.01
3990	2.03
5604	1.96
9937	2.01
12664	2.07

Solution behavior

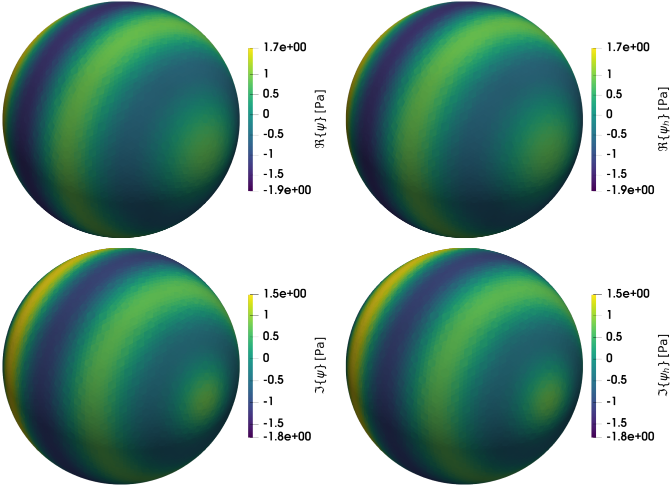


Figure: From right to left: exact and approximate density. From top to bottom: real and imaginary part.

Non-conforming triangular mesh of the sphere

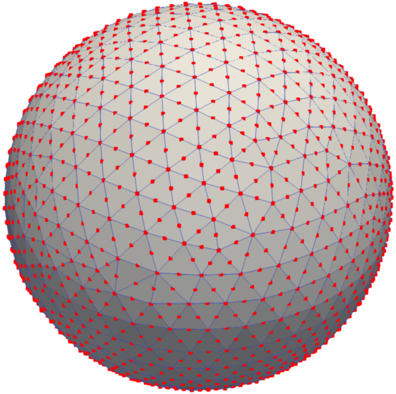


Figure: Triangular mesh of the sphere with 1 hanging node per edge.

$h[m]$	N_{dof}^{6}	N_{dof}^{9}
2.89×10^{-1}	2262	3957
1.11×10^{-1}	8942	15647
6.38×10^{-2}	35218	61630
3.60×10^{-2}	138182	241817

The meshes are not nested.

Convergence behavior

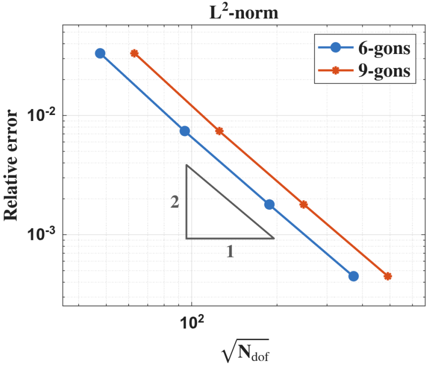


Figure: Convergence behavior of the L^2 relative error.

N_{dof}^6	EOC	N_{dof}^9	EOC
2269	—	3957	—
8942	2.19	15647	2.19
35218	2.09	61630	2.06
138182	2.03	241817	2.03

Conclusions and future works

Conclusions.

We've presented a novel strategy to numerically solve a hypersingular boundary integral equation based on a lowest-order Virtual Element discretization for acoustic scattering problems.

Future works.

- Increase the polynomial degree;
- Tackle time dependent problem (i.e. wave equation);
- Analyze a Virtual Element approximation-based BEM-VEM coupling strategy.