



VEM/BEM coupling for 3D acoustic scattering problems

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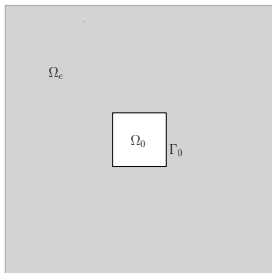


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The model problem - 1

Let Ω_0 be a bounded polyhedral domain in $\mathbb{R}^3$ and define $\Omega_e := \mathbb{R}^3 \setminus \bar{\Omega}_0$. We consider the following **exterior** Dirichlet problem:



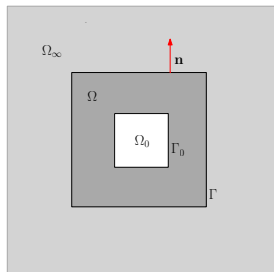
$$\begin{cases} -\Delta u_s - \kappa^2 u_s = f, & \text{in } \Omega_e, \\ u_s = 0, & \text{on } \Gamma_0 := \partial\Omega_0, \\ \lim_{r \rightarrow \infty} r(\partial_r - i\kappa)u_s = 0, & \end{cases} \quad (1)$$

where u_s is the (unknown) scattered field, $\kappa \in \mathbb{R}^+$ is the wavenumber and $r = \|\mathbf{x}\|$, with $\mathbf{x} = (x_1, x_2, x_3)^T$.



The model problem - 2

Now we introduce an artificial boundary Γ which allows to decompose the unbounded domain $\mathbb{R}^3 \setminus \bar{\Omega}_0$ into a finite computational domain Ω and an infinite residual one Ω_∞ .



$$\left\{ \begin{array}{ll} -\Delta u - \kappa^2 u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_0, \\ \gamma^- u = \gamma^+ u_\infty, & \text{on } \Gamma, \\ \partial_{\mathbf{n}}^- u = \partial_{\mathbf{n}}^+ u_\infty, & \text{on } \Gamma, \\ -\Delta u_\infty - \kappa^2 u_\infty = 0, & \text{in } \Omega_\infty, \\ \lim_{r \rightarrow \infty} r(\partial_r - i\kappa)u_\infty = 0, & \end{array} \right.$$

where u is the restriction of u_s to Ω , u_∞ is the restriction of u_s to Ω_∞ , $\gamma^\pm$ is the exterior/interior trace operator and $\partial_{\mathbf{n}}^\pm v := \mathbf{n} \cdot \gamma^\pm \nabla v$.



VEM/BEM coupling

VEM: Performs really well in bounded domains with complex geometries.

BEM: Reformulates the differential problem defined in an unbounded domain into an integral equation on the boundary.

Why don't we take advantage from both approaches?



Kirchhoff's representation formula - 1

It is well-known that u_∞ admits the following integral representation:

$$u_\infty(\mathbf{x}) = \int_\Gamma \frac{\partial G_\kappa(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} \psi(\mathbf{y}) d\mathbf{y} - \int_\Gamma G_\kappa(\mathbf{x}, \mathbf{y}) \lambda(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \Omega_\infty$$

where

$$G_\kappa(\mathbf{x}, \mathbf{y}) = \frac{e^{i\kappa\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|}$$

is the fundamental solution of the homogeneous Helmholtz equation and

$$\psi := \gamma^+ u_\infty = \gamma^- u, \quad \lambda := \partial_{\mathbf{n}}^+ u_\infty = \partial_{\mathbf{n}}^- u \quad \text{on } \Gamma.$$



Boundary Integral Operators (BIOs)

We introduce the boundary integral operators

$$\begin{aligned} V_\kappa \lambda(\mathbf{x}) &:= \int_\Gamma G_\kappa(\mathbf{x}, \mathbf{y}) \lambda(\mathbf{y}) d\mathbf{y}, & K'_\kappa \lambda(\mathbf{x}) &:= \int_\Gamma \frac{\partial G_\kappa(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_\mathbf{x}} \psi(\mathbf{y}) d\mathbf{y}, \\ K_\kappa \psi(\mathbf{x}) &:= \int_\Gamma \frac{\partial G_\kappa(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} \psi(\mathbf{y}) d\mathbf{y}, & W_\kappa \lambda(\mathbf{x}) &:= -\frac{\partial}{\partial \mathbf{n}_\mathbf{x}} \int_\Gamma \frac{\partial G_\kappa(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} \psi(\mathbf{y}) d\mathbf{y}, \end{aligned}$$

with $\mathbf{x} \in \Gamma$. From the integral representation, we obtain the two following BIEs¹:

$$\left(\frac{1}{2} \mathbf{I} - K_\kappa \right) \psi + V_\kappa \lambda = 0, \quad \text{on } \Gamma, \quad (2)$$

$$-W_\kappa \psi + \left(\frac{1}{2} \mathbf{I} - K'_\kappa \right) \lambda = \lambda, \quad \text{on } \Gamma. \quad (3)$$

¹S.A. Sauter and C. Schwab, *Boundary Element Methods*, SSCM, 39. Springer-Verlag, Berlin, 2011.



The model problem - 3

Employing the two boundary integral equations obtained, the problem now becomes

$$\left\{ \begin{array}{ll} -\Delta u - \kappa^2 u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_0, \\ \gamma^- u = \gamma^+ u_\infty, & \text{on } \Gamma, \\ \partial_{\mathbf{n}}^- u = \partial_{\mathbf{n}}^+ u_\infty, & \text{on } \Gamma, \\ \left(\frac{1}{2}\mathbf{I} - \mathbf{K}_\kappa\right) \psi + \mathbf{V}_\kappa \lambda = 0, & \text{on } \Gamma, \\ -\mathbf{W}_\kappa \psi + \left(\frac{1}{2}\mathbf{I} - \mathbf{K}'_\kappa\right) \lambda = \lambda, & \text{on } \Gamma. \end{array} \right.$$



Variational formulation - 1

Define $X := H_{0,\Gamma_0}^1(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_0} = 0\}$ and $\mathbb{X} := X \times H_0^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.
We seek for $(u, \psi, \lambda) \in \mathbb{X}$ such that

$$a(u, v) - \kappa^2 m(u, v) - \langle \lambda, \gamma v - \varphi \rangle_\Gamma - \langle \lambda, \varphi \rangle_\Gamma + \langle \mu, \gamma u - \psi \rangle_\Gamma = m(f, v)$$

is satisfied for all $(v, \varphi, \mu) \in \mathbb{X}$, where

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v, \quad m(u, v) := \int_{\Omega} u v, \quad \langle \cdot, \cdot \rangle_\Gamma = \text{duality product } H^{-1/2}(\Gamma) - H^{1/2}(\Gamma).$$

Inserting the **boundary equations (2)-(3)** into the **fourth term** of the left hand side, we obtain the following variational formulation:



Variational formulation - 2

Find all $(u, \psi, \lambda) \in \mathbb{X}$ such that

$$\mathbb{A}_\kappa((u, \psi, \lambda), (v, \varphi, \mu)) = \mathbb{F}(v, \varphi, \mu) := m(f, v), \quad \forall (v, \varphi, \mu) \in \mathbb{X} \quad (4)$$

for all $(v, \varphi, \mu) \in \mathbb{X}$,

where

$$\begin{aligned} \mathbb{A}_\kappa((u, \psi, \lambda), (v, \varphi, \mu)) &:= \mathbf{A}_\kappa((u, \psi, \lambda), (v, \varphi, \mu)) + \langle \mathbf{W}_\kappa \psi, \varphi \rangle_\Gamma \\ &\quad + \langle \mu, \mathbf{V}_\kappa \lambda \rangle_\Gamma - \langle \lambda, (\tfrac{1}{2}\mathbf{I} - \mathbf{K}_\kappa) \varphi \rangle_\Gamma + \langle \mu, (\tfrac{1}{2}\mathbf{I} - \mathbf{K}_\kappa) \psi \rangle_\Gamma, \end{aligned}$$

and

$$\mathbf{A}_\kappa((u, \psi, \lambda), (v, \varphi, \mu)) := a(u, v) - \kappa^2 m(u, v) - \langle \lambda, \gamma v - \varphi \rangle_\Gamma + \langle \mu, \gamma u - \psi \rangle_\Gamma.$$



Well-posedness of the continuous problem

Theorem (¹)

*Assume that κ^2 is not an eigenvalue of the Laplacian in Ω with a Dirichlet boundary condition on Γ . Then, problem (4) is well posed.
In particular, it admits a unique solution.*

¹G. N. Gatica, S. Meddahi, *Coupling of virtual element and boundary element methods for the solution of acoustic scattering problems*, J. Numer. Math 28(2020), pp. 223-245.



Discrete setting

Let

- $\{\mathcal{T}_h\}$ be a family of polyhedral decompositions of $\overline{\Omega}$ such that

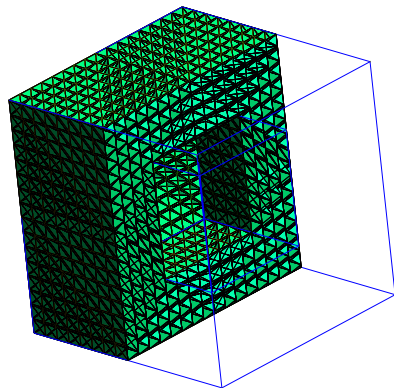
- $\bigcup_{E \in \mathcal{T}_h} \overline{E} = \overline{\Omega},$

- $f|_E \in H^k(E).$

Each element $E \in \mathcal{T}_h$ has diameter $h_E \leq h$;

- $\mathcal{F}_h$ the set of faces of $\mathcal{T}_h$ contained in Γ .

Moreover, we suppose that the mesh satisfies the usual assumptions required by the VEM¹.



¹B. Ahmad, A. Alsaedi, F. Brezzi, L.D. Marini, A. Russo, *Equivalent projectors for virtual element methods*, Comput. Math. Appl. 66 (2013), no. 3, 376–391



VEM spaces - 1

We introduce the following local VEM spaces

$$\begin{aligned} X_h^k(F) &:= \{v \in H^1(F) : v|_e \in \mathbb{P}_k(e), \forall e \subseteq \partial F, \Delta v \in \mathbb{P}_k(F), \Pi_k^F v - \Pi_k^{\nabla, F} v \in \mathbb{P}_{k-2}(F)\}, \\ X_h^k(\partial E) &:= \{v \in C^0(\partial E) : v|_F \in X_h^k(F), \forall F \subseteq \partial E\}, \\ W_h^k(E) &:= \{v \in H^1(E) : v|_{\partial E} \in X_h^k(\partial E), \Delta v \in \mathbb{P}_k(E), \Pi_k^E v - \Pi_k^{\nabla, E} v \in \mathbb{P}_{k-2}(E)\}. \end{aligned}$$

The degrees of freedom of the space $W_h^k(E)$ are:

- the values at the vertices of E ,
- the moments of order $\leq k - 2$ on the edges of E ,
- the moments of order $\leq k - 2$ on the faces of E ,
- the moments of order $\leq k - 2$ on E .

Then the global VEM space is $W_h^k := \{v \in X : v|_E \in W_h^k(E), \forall E \in \mathcal{T}_h\}$.

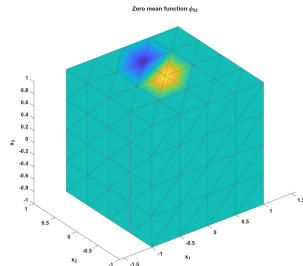
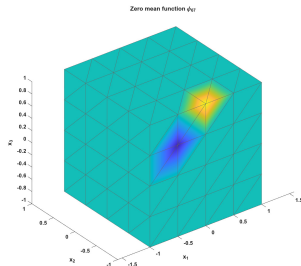
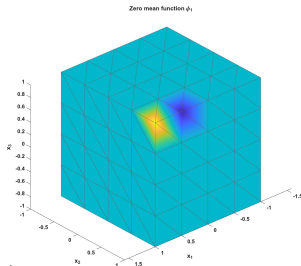


Discrete boundary spaces

We now introduce the simplicial mesh $\mathfrak{F}_h$ of Γ obtained by subdividing each face $F \in \mathcal{F}_h$ into the set of triangles that arise after joining each vertex of F with the midpoint of the disc with respect to which F is star-shaped.

$$\Lambda_h^{k-1} := \{\mu_h \in L^2(\Gamma) : \mu_h|_T \in \mathbb{P}_{k-1}(T), \forall T \in \mathfrak{F}_h\},$$

$$\Psi_h^k := \{\varphi_h \in \mathcal{C}^0(\Gamma) : \varphi_h|_T \in \mathbb{P}_k(T), \forall T \in \mathfrak{F}_h\} \cap H_0^{1/2}(\Gamma).$$





A VEM/BEM scheme

Find $(u_h, \psi_h, \lambda_h) \in \mathbb{X}_h := W_h^k \times \Psi_h^k \times \Lambda_h^{k-1}$ such that

$$\mathbb{A}_{h,\kappa}((u_h, \psi_h, \lambda_h), (v_h, \varphi_h, \mu_h)) = \mathbb{F}_h(v_h, \varphi_h, \mu_h) := m(\Pi_{k-1}^{\mathcal{T}} f, v_h) \quad (5)$$

for all $(v_h, \varphi_h, \mu_h) \in \mathbb{X}_h$.

The operator $\Pi_{k-1}^{\mathcal{T}}$ is the global L^2 projection operator on the space of polynomials of degree $\leq k-1$.



A VEM/BEM scheme

where

$$\begin{aligned} \mathbb{A}_{h,\kappa}((\mathbf{u}_h, \psi_h, \lambda_h), (\mathbf{v}_h, \varphi_h, \mu_h)) &:= \mathbf{A}_{h,\kappa}((\mathbf{u}_h, \psi_h, \lambda_h), (\mathbf{v}_h, \varphi_h, \mu_h)) \\ &+ \langle \mathbf{W}_\kappa \psi_h, \varphi_h \rangle + \langle \mu_h, \mathbf{V}_\kappa \lambda_h \rangle - \langle \lambda_h, (\tfrac{1}{2}\mathbf{I} - \mathbf{K}_\kappa) \varphi_h \rangle + \langle \mu_h, (\tfrac{1}{2}\mathbf{I} - \mathbf{K}_\kappa) \psi_h \rangle, \end{aligned}$$

and

$$\begin{aligned} \mathbf{A}_{h,\kappa}((\mathbf{u}_h, \psi_h, \lambda_h), (\mathbf{v}_h, \varphi_h, \mu_h)) &:= \mathbf{a}_h(\mathbf{u}_h, \mathbf{v}_h) - \kappa^2 \sum_{E \in \mathcal{T}_h} m^E(\Pi_{k-1}^E \mathbf{u}_h, \Pi_{k-1}^E \mathbf{v}_h) \\ &- \sum_{F \in \mathcal{F}_h} \int_F \lambda_h \Pi_{k-1}^F(\gamma \mathbf{v}_h - \varphi_h) + \sum_{F \in \mathcal{F}_h} \int_F \mu_h \Pi_{k-1}^F(\gamma \mathbf{u}_h - \psi_h). \end{aligned}$$



The bilinear form a_h

Denote by $\mathcal{E}(E)$ and $\mathcal{F}(E)$ the set of edges and faces, respectively, of a given $E \in \mathcal{T}_h$.

$$a_h(v, z) := \sum_{E \in \mathcal{T}_h} a_h^E(v, z), \quad \forall v, z \in W_h^k,$$

$$a_h^E(v, z) := \int_E (\Pi_{k-1}^E \nabla v) \cdot (\Pi_{k-1}^E \nabla z) + S_h^E(v - \Pi_k^{\nabla, E} v, z - \Pi_k^{\nabla, E} z),$$

$$S_h^E(v, z) := \sum_{e \in \mathcal{E}(E)} \int_e (\Pi_k^e v) (\Pi_k^e z) + h_E^{-1} \sum_{F \in \mathcal{F}(E)} \int_e (\Pi_{k-2}^F v) (\Pi_{k-2}^F z).$$



Order of convergence

Theorem (¹)

Assume that κ^2 is not an eigenvalue of the laplacian in Ω with a Dirichlet boundary condition on Γ . Let $u \in X \cap \prod_{i=1}^N H^{k+1}(\Omega_i)$ and $f \in \prod_{i=1}^N H^k(\Omega_i)$. Then there exist two constants $C > 0$, independent of h , and $h_0 > 0$ such that for any $h \leq h_0$ the following estimate holds

$$\|(u, \psi, \lambda) - (u_h, \psi_h, \lambda_h)\| \lesssim C h^k \sum_{E \in \mathcal{T}_h} \left(\|u\|_{k+1,E} + \|f\|_{k,E} \right).$$

¹G. N. Gatica, S. Meddahi, *Coupling of virtual element and boundary element methods for the solution of acoustic scattering problems*, J. Numer. Math 28(2020), pp. 223-245.



A numerical test

In our numerical tests, we consider the problem:

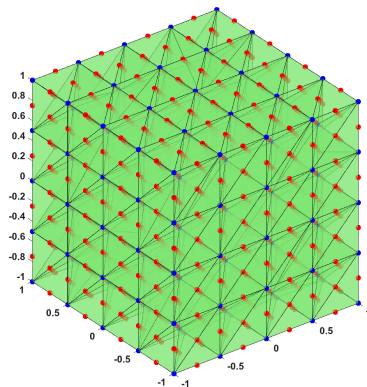
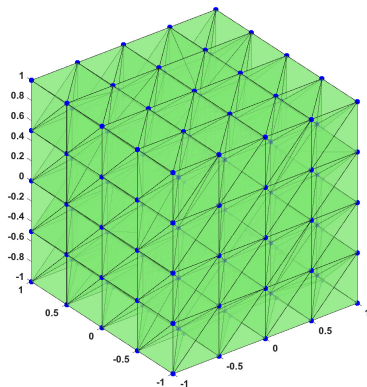
$$\begin{cases} -\Delta u_s - \kappa^2 u_s = 0, & \text{in } \Omega_e, \\ u_s = g, & \text{on } \Gamma_0, \\ \lim_{r \rightarrow \infty} r(\partial_r - i\kappa)u_s = 0, \end{cases} \quad (6)$$

Let $\Omega_0 = [-0.5, 0.5]^3$ and $\Gamma = \partial([-1, 1]^3)$. We choose the Dirichlet datum g on Γ_0 such that the exact solution of problem (6) is given by

$$u(\mathbf{x}) = \frac{e^{i\kappa\|\mathbf{x}\|}}{\|\mathbf{x}\|^2} \left(1 + \frac{i}{\kappa\|\mathbf{x}\|} \right) x_3, \quad \text{for } \mathbf{x} \in \Omega.$$

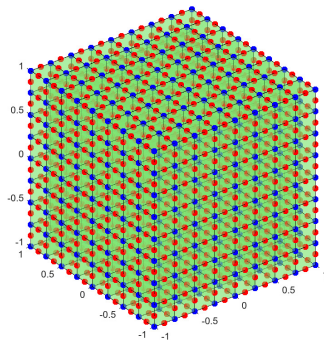
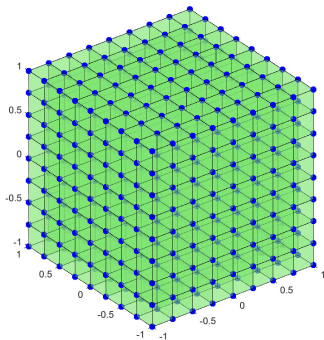


Tetrahedral mesh for degree $k = 1$ and $k = 2$





Hexahedral mesh for degree $k = 1$ and $k = 2$



¹S. Berrone, A. Borio, G. Teora, F. Vicini, *POLYDIM: A C++ library for POLYtopal Discretization Methods*, Comput. Phys. Commun., 320 (2026)



Convergence results, $\kappa = 1$

Tetrahedral mesh

h	$H^1(k=1)$	EOC	$H^1(k=2)$	EOC
1.4142×10^0	8.1269×10^{-1}	-	4.3321×10^{-1}	-
8.2916×10^{-1}	6.3967×10^{-1}	0.35	1.9509×10^{-1}	1.15
4.1458×10^{-1}	3.7085×10^{-1}	0.79	6.1768×10^{-2}	1.66
2.0729×10^{-1}	1.9841×10^{-1}	0.90	1.7504×10^{-2}	1.82

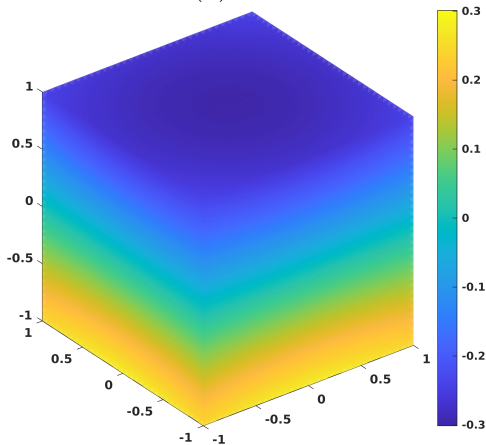
Hexahedral mesh

h	$H^1(k=1)$	EOC	$H^1(k=2)$	EOC
8.6603×10^{-1}	6.5534×10^{-1}	-	3.4588×10^{-1}	-
4.3301×10^{-1}	3.7405×10^{-1}	0.81	1.1947×10^{-1}	1.53
2.1651×10^{-1}	1.8972×10^{-1}	0.98	3.3351×10^{-2}	1.84
1.0825×10^{-1}	9.3701×10^{-2}	1.02	8.5576×10^{-3}	1.96

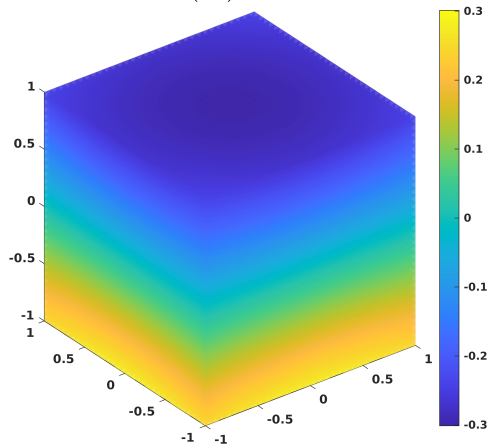


Behavior of the exact and approximate solutions, $\kappa = 1$

$\Re(u)$ on Γ



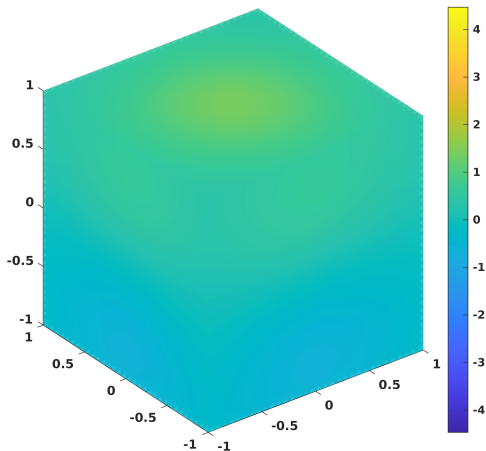
$\Re(u_h)$ on Γ



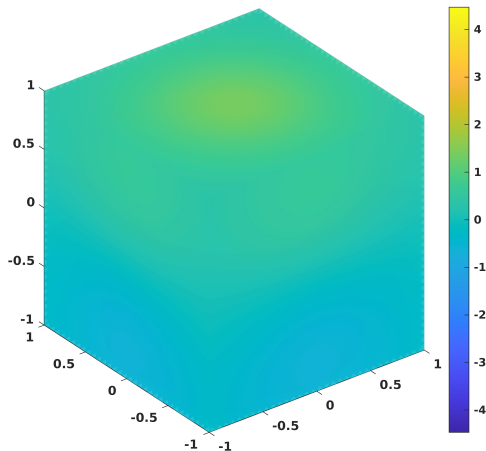


Behavior of the exact and approximate solutions, $\kappa = 1$

$\Im(u)$ on Γ



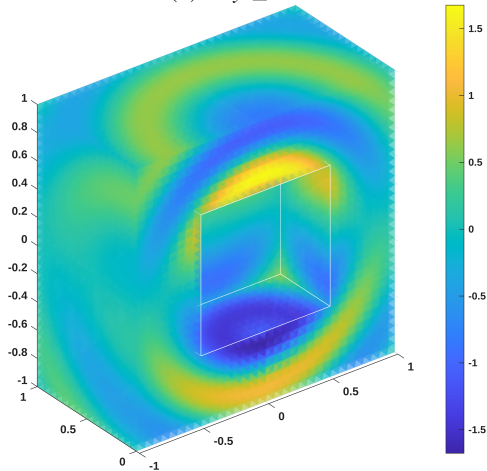
$\Im(u_h)$ on Γ



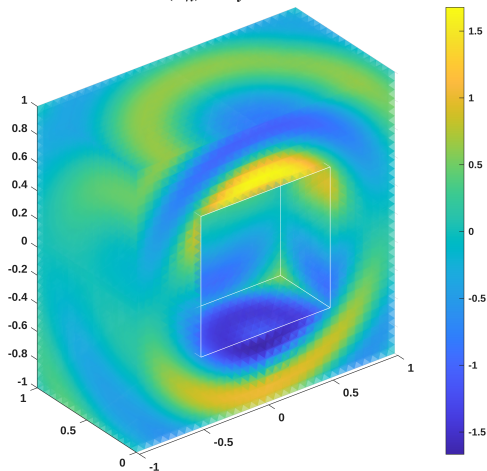


Behavior of the exact and approximate solutions, $\kappa = 10$

$\Re(u)$ on $y \geq 0$



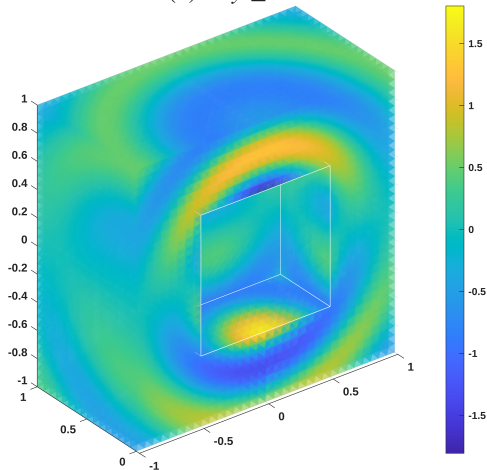
$\Re(u_h)$ on $y \geq 0$



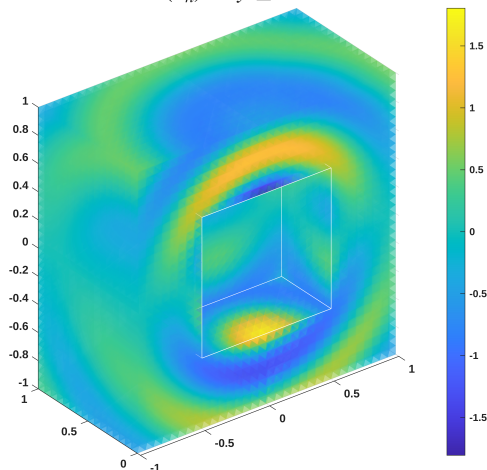


Behavior of the exact and approximate solutions, $\kappa = 10$

$\Im(u)$ on $y \geq 0$



$\Im(u_h)$ on $y \geq 0$





Thank you for your attention!